Filomat 29:5 (2015), 1137–1148
DOI 10.2298/FIL1505137S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Nonlinear $\mathcal{L}$-Fuzzy Stability of k-Cubic Functional Equation

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Abstract. In this paper, we establish the stability result for the k-cubic functional equation

$$2[kf(x+ky)+f(kx-y)] = k(k^2+1)[f(x+y)+f(x-y)]+2(k^4-1)f(y),$$

where k is a real number different from 0 and 1, in the setting of various $\mathcal{L}$-fuzzy normed spaces that in turn generalize a Hyers-Ulam stability result in the framework of classical normed spaces. First we shall prove the stability of k-cubic functional equations in the $\mathcal{L}$-fuzzy normed space under arbitrary t-norm which generalizes previous works. Then we prove the stability of k-cubic functional equations in the non-Archimedean $\mathcal{L}$-fuzzy normed space. We therefore provide a link among different disciplines: fuzzy set theory, lattice theory, non-Archimedean spaces and mathematical analysis.

1. Introduction

The study of stability problems for functional equations is related to a question of Ulam [45] concerning the stability of group homomorphisms and it was affirmatively answered for Banach spaces by Hyers [25]. Subsequently, the result of Hyers was generalized by Aoki [5] for additive mappings and by Th.M. Rassias [34] for linear mappings by considering an unbounded Cauchy difference. The paper [34] of Th.M. Rassias has provided a lot of influence in the development of what we now call Hyers-Ulam-Rassias stability of functional equations. The interested readers for more information on such problems are referred to the works [7–12, 26, 30, 36, 37].

The functional equation

$$2[kf(x+ky)+f(kx-y)] = k(k^2+1)[f(x+y)+f(x-y)]+2(k^4-1)f(y) \tag{1}$$

where k is a real number different from 0 and 1, is said to be the *k-cubic functional equation* which is introduced by the second author.

Note that, if we replace $x=y=0$ in the equation (1), then we get $f(0)=0$. Therefore, setting $x=x$ and $y=0$ in the equation (1), we obtain $f(kx)=(k^3)f(x)$, i.e., $f(x)=kx^3$ is its solution. Every solution of

2010 *Mathematics Subject Classification*. Primary 54E40; Secondary 39B82, 46S50, 46S40.
Keywords. Stability; cubic functional equation; fuzzy normed space, $\mathcal{L}$-fuzzy set.
Received: 14 November 2013; Accepted: 11 February 2014
Communicated by Naseer Shahzad
Research supported by the Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2013053358).
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the k-cubic functional equations is said to be a *cubic mapping*. The stability problem for the cubic functional equations was studied by Jun and Kim [28] for mappings $f : X \longrightarrow Y$, where X is a real normed space and Y is a Banach space. Later, a number of mathematicians worked on the stability of some types of cubic equations [19, 20, 29, 34, 46]. Furthermore, Mirmostafaee, Mirzavaziri and Moslehian [32], Alsina [4], Miheţ and Radu [31] investigated the stability in the settings of fuzzy, probabilistic and random normed spaces.

2. Preliminaries

In this section, we recall some definitions and results which are needed to prove our main results.

A *triangular norm* (shorter *t-norm*) is a binary operation on the unit interval $[0,1]$, i.e., a function $T : [0,1]\times[0,1] \to [0,1]$ such that the following four axioms are satisfied: for all $a,b,c \in [0,1]$,

(1) $T(a,b) = T(b,a)$ (: commutativity);

(2) $T(a,(T(b,c))) = T(T(a,b),c)$ (: associativity);

(3) $T(a,1) = a$ (: boundary condition);

(4) $T(a,b) \le T(a,c)$ whenever $b \le c$ (: monotonicity).

Basic examples are the Łukasiewicz t-norm T_L, $T_L(a,b) = \max\{a+b-1,0\}$ for all $a,b \in [0,1]$ and the t-norms T_P, T_M, T_D, where $T_P(a,b) := ab$, $T_M(a,b) := \min\{a,b\}$,

$$T_D(a,b) := \begin{cases} \min\{a,b\}, & \text{if } \max\{a,b\} = 1; \\ 0, & \text{otherwise.} \end{cases}$$

If T is a t-norm then $x_T^{(n)}$ is defined for all $x \in [0,1]$ and $n \ge 0$ by 1, if $n = 0$ and $T(x_T^{(n-1)}, x)$, if $n \ge 1$. A t-norm T is said to be *of Hadžić-type* (we denote by $T \in \mathcal{H}$) if the family $\{x_T^{(n)}\}$ is equicontinuous at $x = 1$ (cf. [22]).

Other important triangular norms are (see [23]):

(1) The *Sugeno-Weber family* $\{T_\lambda^{SW}\}_{\lambda\in[-1,\infty]}$ is defined by $T_{-1}^{SW} = T_D$, $T_\infty^{SW} = T_P$ and

$$T_\lambda^{SW}(x,y) = \max\left\{0, \frac{x+y-1+\lambda xy}{1+\lambda}\right\}$$

if $\lambda \in (-1,\infty)$.

(2) The *Domby family* $\{T_\lambda^D\}_{\lambda\in[0,\infty]}$ defined by T_D, if $\lambda = 0$, T_M, if $\lambda = \infty$ and

$$T_\lambda^D(x,y) = \frac{1}{1+((\frac{1-x}{x})^\lambda + (\frac{1-y}{y})^\lambda)^{1/\lambda}}$$

if $\lambda \in (0,\infty)$.

(3) The *Aczel-Alsina family* $\{T_\lambda^{AA}\}_{\lambda\in[0,\infty]}$ defined by T_D, if $\lambda = 0$, T_M, if $\lambda = \infty$ and

$$T_\lambda^{AA}(x,y) = e^{-(|\log x|^\lambda + |\log y|^\lambda)^{1/\lambda}}$$

if $\lambda \in (0,\infty)$.

A t-norm T can be extended (by associativity) in a unique way to an n-array operation taking for $(x_1,\cdots,x_n) \in [0,1]^n$ the value $T(x_1,\cdots,x_n)$ defined by

$$\mathrm{T}_{i=1}^0 x_i = 1, \mathrm{T}_{i=1}^n x_i = T(\mathrm{T}_{i=1}^{n-1} x_i, x_n) = T(x_1,\cdots,x_n).$$

The t-norm T can also be extended to a countable operation taking, for any sequence $\{x_n\}$ in $[0,1]$, the value

$$\mathrm{T}_{i=1}^\infty x_i = \lim_{n\to\infty} \mathrm{T}_{i=1}^n x_i. \tag{2}$$

The limit on the right side of (2) exists since the sequence $\{\mathrm{T}_{i=1}^{n} x_i\}_{n\geq 1}$ is non-increasing and bounded from below.

Proposition 2.1. ([23]) (1) *For any $T \geq T_L$, the following implication holds:*

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^{\infty} x_{n+i} = 1 \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty}(1-x_n) < \infty.$$

(2) *If T is of Hadžić-type, then*

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^{\infty} x_{n+i} = 1$$

for any sequence $\{x_n\}$ in $[0,1]$ such that $\lim_{n\to\infty} x_n = 1$.

(3) *If $T \in \{T_\lambda^{AA}\}_{\lambda\in(0,\infty)} \cup \{T_\lambda^{D}\}_{\lambda\in(0,\infty)}$, then*

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^{\infty} x_{n+i} = 1 \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty}(1-x_n)^{\alpha} < \infty.$$

(4) *If $T \in \{T_\lambda^{SW}\}_{\lambda\in[-1,\infty)}$, then*

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^{\infty} x_{n+i} = 1 \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty}(1-x_n) < \infty.$$

3. $\mathcal{L}$-Fuzzy Normed Spaces

The theory of fuzzy sets was introduced by Zadeh in 1965 [47]. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories. Among other fields, a progressive development has been made in the field of fuzzy topology [3, 15–17, 24, 39]. One of the problems in $\mathcal{L}$-fuzzy topology is to obtain an appropriate concept of $\mathcal{L}$-fuzzy metric spaces and $\mathcal{L}$-fuzzy normed spaces. Saadati and Park [40], respectively, introduced and studied a notion of intuitionistic fuzzy metric (normed) spaces and then Deschrijver, Saadati and et. al. generalized the concept of intuitionistic fuzzy metric (normed) spaces and introduced and studied a notion of $\mathcal{L}$-fuzzy metric spaces and $\mathcal{L}$-fuzzy normed spaces (se [13, 41]).

In this section, we give some definitions and related lemmas for our main results.

Definition 3.1. ([18]) Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice and U be a nonempty set called the *universe*. A $\mathcal{L}$-fuzzy set $\mathcal{A}$ on U is defined as a mapping $\mathcal{A} : U \longrightarrow L$. For each u in U, $\mathcal{A}(u)$ represents the *degree* (in L) to which u satisfies $\mathcal{A}$.

Lemma 3.2. ([14]) *Consider the set L^* and operation $\leq_{L^*}$ defined by:*

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \quad \Longleftrightarrow \quad x_1 \leq y_1 \text{ and } x_2 \geq y_2$$

for all $(x_1, x_2), (y_1, y_2) \in L^*$. Then $(L^*, \leq_{L^*})$ is a complete lattice.

Definition 3.3. ([6]) An *intuitionistic fuzzy set* $\mathcal{A}_{\zeta,\eta}$ on a universe U is an object $\mathcal{A}_{\zeta,\eta} = \{(\zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u)) : u \in U\}$, where, for all $u \in U$, $\zeta_{\mathcal{A}}(u) \in [0,1]$ and $\eta_{\mathcal{A}}(u) \in [0,1]$ are called the *membership degree* and the *non-membership degree*, respectively, of u in $\mathcal{A}_{\zeta,\eta}$ and, furthermore, satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.

In section 2, we presented the classical definition of t-norm, which can be easily extended to any lattice $\mathcal{L} = (L, \leq_L)$. Define first $0_{\mathcal{L}} = \inf L$ and $1_{\mathcal{L}} = \sup L$.

Definition 3.4. A *triangular norm* (shortly, *t-norm*) on $\mathcal{L}$ is a mapping $\mathcal{T} : L^2 \to L$ satisfying the following conditions:
(1) $\mathcal{T}(x, 1_{\mathcal{L}}) = x$ for all $x \in L$ (: boundary condition);
(2) $\mathcal{T}(x, y) = \mathcal{T}(y, x)$ for all $(x, y) \in L^2$ (: commutativity);
(3) $\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z)$ for all $(x, y, z) \in L^3$ (: associativity);
(4) $x \leq_L x'$ and $y \leq_L y' \Rightarrow \mathcal{T}(x, y) \leq_L \mathcal{T}(x', y')$ for all $(x, x', y, y') \in L^4$ (: monotonicity).

A t-norm can also be defined recursively as an $(n + 1)$-array operation ($n \geq 1$) by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n(x_{(1)}, \cdots, x_{(n+1)}) = \mathcal{T}(\mathcal{T}^{n-1}(x_{(1)}, \cdots, x_{(n)}), x_{(n+1)})$$

for all $n \geq 2$ and $x_{(i)} \in L$.

The t-norm $\mathbf{M}$ defined by

$$\mathbf{M}(x, y) = \begin{cases} x & \text{if } x \leq_L y, \\ y & \text{if } y \leq_L x, \end{cases}$$

is a continuous t-norm.

Definition 3.5. *A t-norm $\mathcal{T}$ on L^* is said to be* t-representable *if there exist a t-norm T and a t-conorm S on* $[0, 1]$ *such that*

$$\mathcal{T}(x, y) = (T(x_1, y_1), S(x_2, y_2))$$

for all $x = (x_1, x_2), y = (y_1, y_2) \in L^*$.

Definition 3.6. A *negation* on $\mathcal{L}$ is any strictly decreasing mapping $\mathcal{N} : L \to L$ satisfying $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$. If $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in L$, then $\mathcal{N}$ is called an *involutive negation*.

In this paper, let $\mathcal{N} : L \to L$ be a given mapping. The negation N_s on $([0, 1], \leq)$ defined as $N_s(x) = 1 - x$ for all $x \in [0, 1]$ is called the *standard negation* on $([0, 1], \leq)$.

Definition 3.7. The 3-tuple $(V, \mathcal{P}, \mathcal{T})$ is said to be a *$\mathcal{L}$-fuzzy normed space* if V is a vector space, $\mathcal{T}$ is a continuous t-norm on $\mathcal{L}$ and $\mathcal{P}$ is a $\mathcal{L}$-fuzzy set on $V \times (0, +\infty)$ satisfying the following conditions: for all $x, y \in V$ and $t, s \in (0, +\infty)$,
(1) $0_{\mathcal{L}} <_L \mathcal{P}(x, t)$;
(2) $\mathcal{P}(x, t) = 1_{\mathcal{L}}$ if and only if $x = 0$;
(3) $\mathcal{P}(\alpha x, t) = \mathcal{P}(x, \frac{t}{|\alpha|})$ for all $\alpha \neq 0$;
(4) $\mathcal{T}(\mathcal{P}(x, t), \mathcal{P}(y, s)) \leq_L \mathcal{P}(x + y, t + s)$;
(5) $\mathcal{P}(x, \cdot) : (0, \infty) \to L$ is continuous;
(6) $\lim_{t\to 0} \mathcal{P}(x, t) = 0_{\mathcal{L}}$ and $\lim_{t\to\infty} \mathcal{P}(x, t) = 1_{\mathcal{L}}$.

In this case, $\mathcal{P}$ is called a *$\mathcal{L}$-fuzzy norm*. If $\mathcal{P} = \mathcal{P}_{\mu,\nu}$ is an intuitionistic fuzzy set and the t-norm $\mathcal{T}$ is t-representable, then the 3-tuple $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be an *intuitionistic fuzzy normed space (IFN-space)* (see [40] and [43]).

Definition 3.8. (1) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for any $\varepsilon \in L \setminus \{0_{\mathcal{L}}\}$ and $t > 0$, there exists a positive integer n_0 such that

$$\mathcal{N}(\varepsilon) <_L \mathcal{P}(x_{n+p} - x_n, t)$$

for all $n \geq n_0$ and $p > 0$.

(2) If every Cauchy sequence is convergent, then the $\mathcal{L}$-fuzzy norm is said to be *complete* and the $\mathcal{L}$-fuzzy normed space is called a *$\mathcal{L}$-fuzzy Banach space*, where $\mathcal{N}$ is an involutive negation.

(3) The sequence $\{x_n\}$ is said to be *convergent* to a point $x \in V$ in the $\mathcal{L}$-fuzzy normed space $(V, \mathcal{P}, \mathcal{T})$ (denoted by $x_n \xrightarrow{\mathcal{P}} x$) if $\mathcal{P}(x_n - x, t) \to 1_{\mathcal{L}}$ whenever $n \to +\infty$ for all $t > 0$.

Lemma 3.9. ([43]) *Let $\mathcal{P}$ be a $\mathcal{L}$-fuzzy norm on V. Then*
(1) *$\mathcal{P}(x,t)$ is nondecreasing with respect to t for all $x \in V$.*
(2) *$\mathcal{P}(x-y,t) = \mathcal{P}(y-x,t)$ for all $x,y \in V$ and $t \in (0,+\infty)$.*

Definition 3.10. Let $(V,\mathcal{P},\mathcal{T})$ be a $\mathcal{L}$-fuzzy normed space. For any $t \in (0,+\infty)$, we define the *open ball* $B(x,r,t)$ with center $x \in V$ and radius $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ by

$$B(x,r,t) = \{y \in V : \mathcal{N}(r) <_L \mathcal{P}(x-y,t)\}.$$

4. Stability Result in $\mathcal{L}$-fuzzy Normed Spaces

In this section, we study the stability of functional equations in $\mathcal{L}$-fuzzy normed spaces.

Theorem 4.1. *Let X be a linear space and $(Y,\mathcal{P},\mathcal{T})$ be a complete $\mathcal{L}$-fuzzy normed space. If $f : X \to Y$ is a mapping with $f(0) = 0$ and Q is a $\mathcal{L}$-fuzzy set on $X^2 \times (0,\infty)$ with the following property:*

$$\begin{aligned}\mathcal{P}(2[kf(x+ky) + f(kx-y)] - k(k^2+1)[f(x+y) + f(x-y)] - 2(k^4-1)f(y), t)\\ \geq_L Q\left(x, y, \tfrac{t}{2}\right).\end{aligned} \tag{3}$$

If

$$\mathcal{T}_{i=1}^{\infty}(Q(k^{n+i-1}x, 0, k^{3n+2i+1}t)) = 1_{\mathcal{L}}$$

and

$$\lim_{n\to\infty} Q(k^n x, k^n y, k^{3n} t) = 1_{\mathcal{L}}$$

for all $x,y \in X$ and $t > 0$, then there exists a unique k-cubic mapping $C : X \longrightarrow Y$ such that

$$\mathcal{P}(f(x) - C(x), t) \geq_L \mathcal{T}_{i=1}^{\infty}(Q(k^{i-1}x, 0, k^{2i+2}t)). \tag{4}$$

Proof. We brief the proof because it is similar as the random case [1], and [48] also [2]. Putting $y = 0$ in (3), we have

$$\mathcal{P}\left(\frac{f(kx)}{k^3} - f(x), t\right) \geq_{L^*} Q(x, 0, k^3 t).$$

Therefore, it follows that

$$\mathcal{P}\left(\frac{f(k^{j+1}x)}{k^{3(j+1)}} - \frac{f(k^j x)}{k^{3j}}, \frac{t}{k^{j+1}}\right) \geq_L Q(k^j x, 0, k^{2(j+1)}t)$$

for all $j \geq 1$ and $t > 0$. By the triangle inequality, it follows that

$$\mathcal{P}\Big(\frac{f(k^n x)}{k^{3n}} - f(x), t\Big) \quad \geq_L \quad \mathcal{T}_{i=1}^{n}(Q(k^{i-1}x, 0, k^{2i+2}t)). \tag{5}$$

In order to prove the convergence of the sequence $\left\{\frac{f(k^n x)}{k^{3n}}\right\}$, we replace x with $k^m x$ in (5) to find that, for all $m,n > 0$,

$$\mathcal{P}\Big(\frac{f(k^{n+m}x)}{k^{3(n+m)}} - \frac{f(k^m x)}{k^{3m}}, t\Big) \geq_L \mathcal{T}_{i=1}^{n}(Q(k^{i+m-1}x, 0, k^{2i+3m+2}t)).$$

Since the right hand side of the inequality tends to $1_{\mathcal{L}}$ as m tends to ∞, the sequence $\left\{\frac{f(k^n x)}{k^{3n}}\right\}$ is a Cauchy sequence. Thus we may define $C(x) = \lim_{n\to\infty} \frac{f(k^n x)}{k^{3n}}$ for all $x \in X$. Replacing x,y with $k^n x$ and $k^n y$, respectively, in (3), it follows that C is a k-cubic mapping. To prove (4), take the limit as $n \to \infty$ in (5).

To prove the uniqueness of the k-cubic mapping C subject to (4), let us assume that there exists another k-cubic mapping C' which satisfies (4). Obviously, we have $C(k^n x) = k^{3n}C(x)$ and $C'(k^n x) = k^{3n}C'(x)$ for all $x \in X$ and $n \geq 1$. Hence it follows from (4) that

$$\begin{aligned}
&\mathcal{P}\Big(C(x) - C'(x), t\Big) \\
&\geq_L \mathcal{P}\Big(C(k^n x) - C'(k^n x), k3^{3n}t\Big) \\
&\geq_L \mathcal{T}\Big(\mathcal{P}\Big(C(k^n x) - f(k^n x), k^{3n-1}t\Big), \mathcal{P}\Big(f(k^n x) - C'(k^n x), 2^{3n-1}t\Big)\Big) \\
&\geq_L \mathcal{T}\Big(\mathcal{T}_{i=1}^{\infty}(Q(k^{n+i-1}x, 0, k^{3n+2i+1}t)), \mathcal{T}_{i=1}^{\infty}(Q(k^{n+i-1}x, 0, k^{3n+2i+1}t))\Big) \\
&= \mathcal{T}(1_{\mathcal{L}}, 1_{\mathcal{L}}) = 1_{\mathcal{L}}
\end{aligned}$$

for all $x \in X$. This proves the uniqueness of C. This completes the proof.

Corollary 4.2. *Let $(X, \mathcal{P}'_{\mu',\nu'}, \mathcal{T})$ be IFN-space and $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ be a complete IFN-space. Let $f : X \longrightarrow Y$ be a mapping such that*

$$\begin{aligned}
&\mathcal{P}_{\mu,\nu}(2[kf(x + ky) + f(kx - y)] - k(k^2 + 1)[f(x + y) + f(x - y)] - 2(k^4 - 1)f(y), t) \\
\geq_{L^*} \quad &\mathcal{P}'_{\mu',\nu'}\left(x + y, \frac{t}{2}\right)
\end{aligned}$$

for all $t > 0$ in which

$$\lim_{n\longrightarrow\infty} \mathcal{T}_{i=1}^{\infty}(\mathcal{P}'_{\mu',\nu'}(k^{n+i-1}x, k^{3n+2i+1}t)) = 1_{L^*}$$

for all $x, y \in X$. Then there exists a unique k-cubic mapping $C : X \longrightarrow Y$ such that

$$\mathcal{P}_{\mu,\nu}(f(x) - C(x), t) \geq_{L^*} \mathcal{T}_{i=1}^{\infty}(\mathcal{P}'_{\mu',\nu'}(k^{i-1}x, k^{2i+2}t)).$$

Proof. In Theorem 4.1, put $Q(x, y, t) = \mathcal{P}'_{\mu',\nu'}(x + y, t)$. Therefore, all the conditions of Theorem 4.1 hold and so there exists a unique k-cubic mapping $C : X \longrightarrow Y$ such that

$$\mathcal{P}_{\mu,\nu}(f(x) - C(x), t) \geq_{L^*} \mathcal{T}_{i=1}^{\infty}(\mathcal{P}'_{\mu',\nu'}(k^{i-1}x, k^{2i+2}t)).$$

Now, we give one example to illustrate the main results of Theorem 4.1, as follows:

Example 4.3. Let $(X, \|\cdot\|)$ be a Banach algebra space, $(X, \mathcal{P}_{\mu,\nu}, \mathbf{M})$ be IFN-space in which

$$\mathcal{P}_{\mu,\nu}(x, t) = \Big(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|}\Big)$$

and $(Y, \mathcal{P}_{\mu,\nu}, \mathbf{M})$ be a complete IFN-space for all $x \in X$. Define a mapping $f : X \longrightarrow Y$ by $f(x) = x^3 + x_0$, where x_0 is a unit vector in X. A straightforward computation shows that

$$\begin{aligned}
&\mathcal{P}_{\mu,\nu}(2[kf(x + ky) + f(kx - y)] - k(k^2 + 1)[f(x + y) + f(x - y)] - 2(k^4 - 1)f(y), t) \\
\geq_{L^*} \quad &\mathcal{P}_{\mu,\nu}\left(x + y, \frac{t}{2}\right)
\end{aligned}$$

for all $t > 0$. Also, we have

$$\begin{aligned}
\lim_{n\longrightarrow\infty} \mathbf{M}_{i=1}^{\infty}(\mathcal{P}_{\mu,\nu}(k^{n+i-1}x, k^{3n+2i+1}t)) &= \lim_{n\longrightarrow\infty}\lim_{m\longrightarrow\infty} \mathbf{M}_{i=1}^{m}(\mathcal{P}_{\mu,\nu}(x, k^{2n+i+2}t)) \\
&= \lim_{n\longrightarrow\infty}\lim_{m\longrightarrow\infty} \mathcal{P}_{\mu,\nu}(x, k^{2n+3}t) \\
&= \lim_{n\longrightarrow\infty} \mathcal{P}_{\mu,\nu}(x, k^{2n+3}t) \\
&= 1_{L^*}.
\end{aligned}$$

Therefore, all the conditions of Theorem 4.1 hold and so there exists a unique k-cubic mapping $C : X \longrightarrow Y$ such that

$$\mathcal{P}_{\mu,\nu}(f(x) - C(x), t) \geq_{L^*} \mathcal{P}_{\mu,\nu}(x, k^4 t).$$

5. Non-Archimedean $\mathcal{L}$-fuzzy Normed Spaces

In 1897, Hensel [27] introduced a field with a valuation in which does not have the Archimedean property.

Definition 5.1. Let $\mathcal{K}$ be a field. A *non-Archimedean absolute value* on $\mathcal{K}$ is a function $|\cdot| : \mathcal{K} \to [0, +\infty)$ such that, for any $a, b \in \mathcal{K}$,

(1) $|a| \geq 0$ and equality holds if and only if $a = 0$;
(2) $|ab| = |a||b|$;
(3) $|a + b| \leq \max\{|a|, |b|\}$ (: the strict triangle inequality).

Note that $|n| \leq 1$ for each integer n. We always assume, in addition, that $|\cdot|$ is non-trivial, i.e., there exists $a_0 \in \mathcal{K}$ such that $|a_0| \neq 0, 1$.

Definition 5.2. A *non-Archimedean $\mathcal{L}$-fuzzy normed space* is a triple $(V, \mathcal{P}, \mathcal{T})$, where V is a vector space, $\mathcal{T}$ is a continuous t-norm on $\mathcal{L}$ and $\mathcal{P}$ is a $\mathcal{L}$-fuzzy set on $V \times (0, +\infty)$ satisfying the following conditions: for all $x, y \in V$ and $t, s \in (0, +\infty)$,

(1) $0_{\mathcal{L}} <_L \mathcal{P}(x, t)$;
(2) $\mathcal{P}(x, t) = 1_{\mathcal{L}}$ if and only if $x = 0$;
(3) $\mathcal{P}(\alpha x, t) = \mathcal{P}(x, \frac{t}{|\alpha|})$ for all $\alpha \neq 0$;
(4) $\mathcal{T}(\mathcal{P}(x, t), \mathcal{P}(y, s)) \leq_L \mathcal{P}(x + y, \max\{t, s\})$;
(5) $\mathcal{P}(x, \cdot) : (0, \infty) \to L$ is continuous;
(6) $\lim_{t\to 0} \mathcal{P}(x, t) = 0_{\mathcal{L}}$ and $\lim_{t\to\infty} \mathcal{P}(x, t) = 1_{\mathcal{L}}$.

Example 5.3. Let $(X, \|\cdot\|)$ be a non-Archimedean normed linear space. Then the triple $(X, \mathcal{P}, \min)$, where

$$\mathcal{P}(x, t) = \begin{cases} 0, & \text{if } t \leq \|x\|, \\ 1, & \text{if } t > \|x\|, \end{cases}$$

is a non-Archimedean $\mathcal{L}$-fuzzy normed space in which $L = [0, 1]$.

Example 5.4. Let $(X, \|\cdot\|)$ be is a non-Archimedean normed linear space. Denote $\mathcal{T}_M(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and let $\mathcal{P}_{\mu,\nu}$ be the intuitionistic fuzzy set on $X \times (0, +\infty)$ defined as follows:

$$\mathcal{P}_{\mu,\nu}(x, t) = \left(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|}\right)$$

for all $t \in \mathbb{R}^+$. Then $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T}_M)$ is a non-Archimedean intuitionistic fuzzy normed space.

6. $\mathcal{L}$-Fuzzy Hyers-Ulam-Rassias Stability for k-cubic Functional Equations in Non-Archimedean $\mathcal{L}$-fuzzy Normed Space

Let $\mathcal{K}$ be a non-Archimedean field, X be a vector space over $\mathcal{K}$ and $(Y, \mathcal{P}, \mathcal{T})$ be a non-Archimedean $\mathcal{L}$-fuzzy Banach space over $\mathcal{K}$. In this section, we investigate the stability of the k-cubic functional equation (1).

Next, we define a $\mathcal{L}$-fuzzy approximately k-cubic mapping.

Let Ψ be a $\mathcal{L}$-fuzzy set on $X \times X \times [0, \infty)$ such that $\Psi(x, y, \cdot)$ is nondecreasing,

$$\Psi(cx, cx, t) \geq_L \Psi\left(x, x, \frac{t}{|c|}\right)$$

for all $x \in X$ and $c \neq 0$ and

$$\lim_{t\to\infty} \Psi(x, y, t) = 1_{\mathcal{L}}$$

for all $x, y \in X$ and $t > 0$.

Definition 6.1. A mapping $f : X \to Y$ is said to be Ψ-*approximately k-cubic* if

$$\mathcal{P}(2[kf(x+ky)+f(kx-y)]-k(k^2+1)[f(x+y)+f(x-y)]-2(k^4-1)f(y),t) \\ \geq_L \Psi\left(x,y,\tfrac{t}{|2|}\right) \tag{6}$$

for all $x, y \in X$ and $t > 0$.

Here, we assume that $k \neq 0$ in $\mathcal{K}$ (i.e., the characteristic of $\mathcal{K}$ is not k).

Theorem 6.2. *Let $\mathcal{K}$ be a non-Archimedean field, X be a vector space over $\mathcal{K}$ and $(Y, \mathcal{P}, \mathcal{T})$ be a non-Archimedean $\mathcal{L}$-fuzzy Banach space over $\mathcal{K}$. Let $f : X \to Y$ be a Ψ-approximately k-cubic mapping. If there exist $\alpha \in \mathbb{R}$ $(\alpha > 0)$ and an integer $\ell \geq 2$ with $|k^\ell| < \alpha$ and $|k| \neq 1$ such that*

$$\Psi(k^{-\ell}x, k^{-\ell}y, t) \geq_L \Psi(x, y, \alpha t) \tag{7}$$

for all $x \in X$ and $t > 0$ and

$$\lim_{n\to\infty} \mathcal{T}_{j=n}^{\infty}\mathcal{M}\left(x, \frac{\alpha^j t}{|k|^{\ell j}}\right) = 1_{\mathcal{L}}$$

for all $x \in X$ and $t > 0$, then there exists a unique k-cubic mapping $C : X \to Y$ such that

$$\mathcal{P}(f(x) - C(x), t) \geq \mathcal{T}_{i=1}^{\infty}\mathcal{M}\left(x, \frac{\alpha^{i+1} t}{|k|^{\ell i}}\right) \tag{8}$$

for all $x \in X$ and $t > 0$, where

$$\mathcal{M}(x,t) := \mathcal{T}(\Psi(x,0,t), \Psi(kx,0,t), \cdots, \Psi(k^{\ell-1}x, 0, t))$$

for all $x \in X$ and $t > 0$.

Proof. First, we show, by induction on j, that, for all $x \in X$, $t > 0$ and $j \geq 1$,

$$\mathcal{P}(f(k^j x) - k^{3j} f(x), t) \geq_L \mathcal{M}_j(x,t) := \mathcal{T}(\Psi(x,0,t), \cdots, \Psi(k^{j-1}x, 0, t)). \tag{9}$$

Putting $y = 0$ in (6), we obtain

$$\mathcal{P}(f(kx) - k^3 f(x), t) \geq_L \Psi(x, 0, t)$$

for all $x \in X$ and $t > 0$. This proves (9) for $j = 1$. Let (9) hold for some $j > 1$. Replacing y by 0 and x by $k^j x$ in (6), we get

$$\mathcal{P}(f(k^{j+1}x) - k^3 f(k^j x), t) \geq_L \Psi(k^j x, 0, t)$$

for all $x \in X$ and $t > 0$. Since $|k^3| \leq 1$, it follows that

$$\begin{aligned}
&\mathcal{P}(f(k^{j+1}x) - k^{3j+3} f(x), t) \\
&\quad \geq_L \mathcal{T}\Big(\mathcal{P}(f(k^{j+1}x) - k^3 f(k^j x), t), \mathcal{P}(k^3 f(k^j x) - k^{3j+3} f(x), t)\Big) \\
&\quad = \mathcal{T}\Big(\mathcal{P}(f(k^{j+1}x) - k^3 f(3^j x), t), \mathcal{P}\left(f(k^j x) - k^{3j} f(x), \frac{t}{|k^3|}\right)\Big) \\
&\quad \geq_L \mathcal{T}\Big(\mathcal{P}(f(k^{j+1}x) - k^3 f(k^j x), t), \mathcal{P}\left(f(k^j x) - k^{3j} f(x), t\right)\Big) \\
&\quad \geq_L \mathcal{T}(\Psi(k^j x, 0, t), \mathcal{M}_j(x,t)) \\
&\quad = \mathcal{M}_{j+1}(x,t)
\end{aligned}$$

for all $x \in X$ and $t > 0$. Thus (9) holds for all $j \geq 1$. In particular, we have

$$\mathcal{P}(f(k^{\ell}x) - k^{3\ell}f(x), t) \geq_L \mathcal{M}(x, t) \tag{10}$$

for all $x \in X$ and $t > 0$. Replacing x by $k^{-(\ell n+\ell)}x$ in (10) and using the inequality (7), we obtain

$$\mathcal{P}\left(f\left(\frac{x}{k^{\ell n}}\right) - k^{3\ell}f\left(\frac{x}{k^{\ell n+\ell}}\right), t\right) \geq_L \mathcal{M}\left(\frac{x}{k^{\ell n+\ell}}, t\right) \geq_L \mathcal{M}(x, \alpha^{n+1}t)$$

for all $x \in X$, $t > 0$ and $n \geq 0$ and so

$$\mathcal{P}\left((k^{3\ell})^n f\left(\frac{x}{(k^{\ell})^n}\right) - (k^{3\ell})^{n+1} f\left(\frac{x}{(k^{\ell})^{n+1}}\right), t\right) \geq_L \mathcal{M}\left(x, \frac{\alpha^{n+1}}{|(k^{3\ell})^n|}t\right) \geq_L \mathcal{M}\left(x, \frac{\alpha^{n+1}}{|(k^{\ell})^n|}t\right)$$

for all $x \in X$, $t > 0$ and $n \geq 0$. Hence it follow that

$$\begin{aligned}
&\mathcal{P}\left((k^{3\ell})^n f\left(\frac{x}{(k^{\ell})^n}\right) - (k^{3\ell})^{n+p} f\left(\frac{x}{(k^{\ell})^{n+p}}\right), t\right) \\
&\geq_L \mathcal{T}_{j=n}^{n+p}\left(\mathcal{P}((k^{3\ell})^j f\left(\frac{x}{(k^{\ell})^j}\right) - (k^{3\ell})^{j+p} f\left(\frac{x}{(k^{\ell})^{j+p}}\right), t)\right) \\
&\geq_L \mathcal{T}_{j=n}^{n+p}\mathcal{M}\left(x, \frac{\alpha^{j+1}}{|(k^{\ell})^j|}t\right)
\end{aligned}$$

for all $x \in X$, $t > 0$ and $n \geq 0$. Since $\lim_{n\to\infty} \mathcal{T}_{j=n}^{\infty}\mathcal{M}\left(x, \frac{\alpha^{j+1}}{|(k^{\ell})^j|}t\right) = 1_{\mathcal{L}}$ for all $x \in X$ and $t > 0$, $\left\{(k^{3\ell})^n f\left(\frac{x}{(k^{\ell})^n}\right)\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the non-Archimedean $\mathcal{L}$-fuzzy Banach space $(Y, \mathcal{P}, \mathcal{T})$. Hence we can define a mapping $C : X \to Y$ such that

$$\lim_{n\to\infty} \mathcal{P}\left((k^{3\ell})^n f\left(\frac{x}{(k^{\ell})^n}\right) - C(x), t\right) = 1_{\mathcal{L}} \tag{11}$$

for all $x \in X$ and $t > 0$. Next, for all $n \geq 1$, $x \in X$ and $t > 0$, we have

$$\begin{aligned}
&\mathcal{P}\left(f(x) - (k^{3\ell})^n f\left(\frac{x}{(k^{\ell})^n}\right), t\right) \\
&= \mathcal{P}\left(\sum_{i=0}^{n-1}(k^{3\ell})^i f\left(\frac{x}{(k^{\ell})^i}\right) - (k^{3\ell})^{i+1} f\left(\frac{x}{(k^{\ell})^{i+1}}\right), t\right) \\
&\geq_L \mathcal{T}_{i=0}^{n-1}\left(\mathcal{P}((k^{3\ell})^i f\left(\frac{x}{(k^{\ell})^i}\right) - (k^{3\ell})^{i+1} f\left(\frac{x}{(k^{\ell})^{i+1}}\right), t)\right) \\
&\geq_L \mathcal{T}_{i=0}^{n-1}\mathcal{M}\left(x, \frac{\alpha^{i+1}t}{|k^{\ell}|^i}\right)
\end{aligned}$$

and so

$$\begin{aligned}
&\mathcal{P}(f(x) - C(x), t) \\
&\geq_L \mathcal{T}\left(\mathcal{P}(f(x) - (k^{3\ell})^n f\left(\frac{x}{(k^{\ell})^n}\right), t), \mathcal{P}((k^{3\ell})^n f\left(\frac{x}{(k^{\ell})^n}\right) - C(x), t)\right) \\
&\geq_L \mathcal{P}\left(\mathcal{T}_{i=0}^{n-1}\mathcal{M}\left(x, \frac{\alpha^{i+1}t}{|k^{\ell}|^i}\right), \mathcal{P}((k^{3\ell})^n f\left(\frac{x}{(k^{\ell})^n}\right) - C(x), t)\right).
\end{aligned} \tag{12}$$

Taking the limit as $n \to \infty$ in (12), we obtain

$$\mathcal{P}(f(x) - C(x), t) \geq_L \mathcal{T}_{i=1}^{\infty}\mathcal{M}\left(x, \frac{\alpha^{i+1}t}{|k^{\ell}|^i}\right),$$

which proves (8). As $\mathcal{T}$ is continuous, from a well known result in $\mathcal{L}$-fuzzy (probabilistic) normed space (see [44], Chapter 12), it follows that

$$\begin{aligned}&\lim_{n\to\infty}\mathcal{P}((k^{3\ell n})2[kf(k^{-\ell n}(x+ky))+f(k^{-\ell n}(x-ky))]\\&\quad -k^{3\ell n}k(k^2+1)[f((k^{-\ell n}(x+y))+f((k^{-\ell n}(x-y))]-2k^{3\ell n}(k^4-1)f((k^{-\ell n}y),t)\\&\quad =\mathcal{P}(2[kC(x+ky)+C(kx-y)]-k(k^2+1)[C(x+y)+C(x-y)]-2(k^4-1)C(y),t)\end{aligned}$$

for almost all $t>0$.

On the other hand, replacing x,y by $k^{-\ell n}x, k^{-\ell n}y$ in (6) and (7), we get

$$\begin{aligned}&\mathcal{P}((k^{3\ell n})2[kf(k^{-\ell n}(x+ky))+f(k^{-\ell n}(x-ky))]\\&\quad -k^{3\ell n}k(k^2+1)[f((k^{-\ell n}(x+y))+f((k^{-\ell n}(x-y))]-2k^{3\ell n}(k^4-1)f((k^{-\ell n}y),t)\\&\quad \geq_L \Psi\left(k^{-\ell n}x, k^{-\ell n}y, \frac{t}{|k^{3\ell}|^n}\right)\\&\quad \geq_L \Psi\left(x,y,\frac{\alpha^n t}{|k^{\ell}|^n}\right)\end{aligned}$$

for all $x,y\in X$ and $t>0$. Since $\lim_{n\to\infty}\Psi\left(x,y,\frac{\alpha^n t}{|k^\ell|^n}\right)=1_{\mathcal{L}}$, we infer that C is a k-cubic mapping.

Finally, for the uniqueness of C, let $C':X\to Y$ be another k-cubic mapping such that

$$\mathcal{P}(C'(x)-f(x),t)\geq_L \mathcal{M}(x,t)$$

for all $x\in X$ and $t>0$. Then we have, for all $x\in X$ and $t>0$,

$$\begin{aligned}&\mathcal{P}(C(x)-C'(x),t)\\&\quad\geq_L \mathcal{T}\left(\mathcal{P}(C(x)-(k^{3\ell})^n f\left(\frac{x}{(k^\ell)^n}\right),t), \mathcal{P}((k^{3\ell})^n f\left(\frac{x}{(k^\ell)^n}\right)-C'(x),t),t)\right).\end{aligned}$$

Therefore, from (11), we conclude that $C=C'$. This completes the proof.

Corollary 6.3. *Let $\mathcal{K}$ be a non-Archimedean field, X be a vector space over $\mathcal{K}$ and $(Y,\mathcal{P},\mathcal{T})$ be a non-Archimedean $\mathcal{L}$-fuzzy Banach space over $\mathcal{K}$ under a t-norm $\mathcal{T}\in\mathcal{H}$. Let $f:X\to Y$ be a Ψ-approximately cubic mapping. If there exist $\alpha\in\mathbb{R}$ ($\alpha>0$), $|k|\neq 1$ and an integer $\ell\geq 2$ with $|k^\ell|<\alpha$ such that*

$$\Psi(k^{-\ell}x,k^{-\ell}y,t)\geq_L \Psi(x,y,\alpha t)$$

for all $x,y\in X$ and $t>0$, then there exists a unique k-cubic mapping $C:X\to Y$ such that

$$\mathcal{P}(f(x)-C(x),t)\geq_L \mathcal{T}_{i=1}^{\infty}\mathcal{M}\left(x,\frac{\alpha^{i+1}t}{|k|^{\ell i}}\right)$$

for all $x\in X$ and $t>0$, where

$$\mathcal{M}(x,t):=\mathcal{T}(\Psi(x,0,t),\Psi(kx,0,t),\cdots,\Psi(k^{\ell-1}x,0,t))$$

for all $x\in X$ and $t>0$.

Proof. Since

$$\lim_{n\to\infty}\mathcal{M}\left(x,\frac{\alpha^j t}{|k|^{\ell j}}\right)=1_{\mathcal{L}}$$

for all $x\in X$ and $t>0$ and $\mathcal{T}$ is of Hadžić type, it follows from Proposition 2.1 that

$$\lim_{n\to\infty}\mathcal{T}_{j=n}^{\infty}\mathcal{M}\left(x,\frac{\alpha^j t}{|k|^{\ell j}}\right)=1_{\mathcal{L}}$$

for all $x\in X$ and $t>0$. Now, if we apply Theorem 6.2, we get the conclusion. This completes the proof.

Now, we give an example to validate the main result as follows:

Example 6.4. Let $(X, \|\cdot\|)$ be a non-Archimedean Banach space, $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T}_M)$ be non-Archimedean $\mathcal{L}$-fuzzy normed space (intuitionistic fuzzy normed space) in which

$$\mathcal{P}_{\mu,\nu}(x,t) = \Big(\frac{t}{t+\|x\|}, \frac{\|x\|}{t+\|x\|}\Big)$$

for all $x \in X$ and $t > 0$ and $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{T}_M)$ be a complete non-Archimedean $\mathcal{L}$-fuzzy normed space (intuitionistic fuzzy normed space) (see Example 5.4). Define

$$\Psi(x,y,t) = \left(\frac{t}{1+t}, \frac{1}{1+t}\right).$$

It is easy to see that (7) holds for $\alpha = 1$. Also, since

$$\mathcal{M}(x,t) = \left(\frac{t}{1+t}, \frac{1}{1+t}\right),$$

we have

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{T}^{\infty}_{M,j=n} \mathcal{M}\left(x, \frac{\alpha^j t}{|k|^{\ell j}}\right) &= \lim_{n\to\infty}\left(\lim_{m\to\infty} \mathcal{T}^{m}_{M,j=n} \mathcal{M}\left(x, \frac{t}{|k|^{\ell j}}\right)\right) \\
&= \lim_{n\to\infty}\lim_{m\to\infty}\left(\frac{t}{t+|k^\ell|^n}, \frac{|k^\ell|^n}{t+|k^\ell|^n}\right) \\
&= (1,0) = 1_{L^*}
\end{aligned}$$

for all $x \in X$ and $t > 0$. Let $f : X \to Y$ be a Ψ-approximately k-cubic mapping. Therefore, all the conditions of Theorem 6.2 hold and so there exists a unique k-cubic mapping $C : X \longrightarrow Y$ such that

$$\mathcal{P}_{\mu,\nu}(f(x) - C(x), t) \geq_{L^*} \left(\frac{t}{t+|k^\ell|}, \frac{|k^\ell|}{t+|k^\ell|}\right)$$

for all $x \in X$ *and* $t > 0$.

7. Conclusion

We established the Hyers-Ulam-Rassias stability of the k-cubic functional equations (1) in various fuzzy spaces. In section 4, we proved the stability of functional equations (1) in a $\mathcal{L}$-fuzzy normed space under an arbitrary t-norm which is a generalization of [33]. In section 6, we proved the stability of functional equations (1) in a non-Archimedean $\mathcal{L}$-fuzzy normed space. Therefore, we provided a link among three various discipline: fuzzy set theory, lattice theory and mathematical analysis.

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